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MIT 6.859 – Randomness and Computation, with Ronitt Rubinfeld
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Problem Set 3 – Solutions

Problem 1

Let the vertices of the line L_n be named $1, \ldots, n$. And let $R(k)$ be the expected number of random walk steps it takes to move from vertex k to vertex $k+1$, for $1 \leq k < n$. We have that $R(1) = 1$, and:

$$R(k) = \frac{1}{2} + \frac{1}{2}\Big(1 + R(k-1) + R(k)\Big)$$

Solving the recurrence yields $R(k) = 2(k-1) + 1$.

Now, let's devide the cover time of L_n starting from a vertex v into two phases. Phase I starts at $t = 0$ and ends when the walk hits either end-point for the first time. Phase II starts at the end of Phase I and continues until the other endpoint is hit.

It is now easy to see that the durations of both phases are each stochastically dominated by A, the time to hit vertex n, starting from vertex 1. The expected time to do this is simply:

$$A = \sum_{k=1}^{n-1} R(k) = (n-1)^2$$

Therefore the cover time of L_n is $O(n^2)$. On the other hand, a lower bound on the cover time is the expected duration of Phase II, which is exactly equal to A. Therefore the cover time on L_n is in fact $\Theta(n^2)$.

Problem 2

Assuming that the 2-SAT formula on k clauses and n variables has a satisfying assignment, we fix one such assignment δ. Consider a sequence of random variables $X_1, X_2, \ldots$, where X_t equals the number of variables for which the current assignment agrees with δ.

On any given iteration we pick an unsatisfied clause. For any such clause, the assignment of at least one of the involved variables must disagree with δ. Therefore, with probability at least 1/2 we have that $X_{t+1} = X_t + 1$. Further, with probability at most 1/2 we have that $X_{t+1} = X_t - 1$. In the special event that $X_t = 0$, we have that $\mathbf{Pr}[X_{t+1} = X_t + 1 | X_t = 0] = 1$. We are interested in the expected value of the first time τ for which $X_\tau = n$.

This process is stochastically dominated by the process from the previous problem, where we have a random walk on L_n starting from vertex 1, aiming to reach vertrex n. We know that the expected time of this is $\Theta(n^2)$.

This gives us probability at least 1/2 of reaching a satisfying assignment (if it exists). Repeating the entire process 2 times (or equivalently running it for twice as many steps) ensures that the probability of success is at least 3/4, as the probability of failing on both tries is at most 1/4. Note that no restart is needed between the two tries, because the bounds on cover times we use are derived for arbitrary starting states.

Problem 3

Part (a). Begin with n vertices labeled by $[n]$, each with d (directed) outgoing edges labeled by $[d]$, such that the other end-point of each edge is not yet connected. There are nd edges in total, and for each there are n possible connection points. Therefore, there are n^{nd} different directed graphs with out-degree d where both vertices and edges are labeled (i.e. no isomorphisms exist). The number of such graphs dominates the number of edge-labeled, undirected, d-regular graphs, therefore n^{nd} is an upper bound on the number of such graphs.

Part (b). For a fixed undirected, connected d-regular graph G, from lecture we know that the cover time is at most $4|V||E| = 4dn^2 = c$. Consider a random walk (starting from an arbitrary vertex) for w steps, where w is a multiple of c. Treat this walk as w/c disjoint random "sub-walks". The probability that the random walk does not cover the graph is upper-bounded by the probability that no sub-walk covers the graph. The sub-walks are independent of each other, and according to lemma from class each sub-walk covers the graph with probability at least 1/2. Therefore, the probability that the graph is not covered is upper-bounded by the probability that w/c fair coin tosses all turn heads, which is $2^{-w/c}$.

Let X_G be an indicator random variable for the event that the walk does not cover G. Set $X = \sum_G X_G$, and observe that the event that the random walk is universal is equivalent to the event $X = 0$. By linearity of expectation we have that $\mathbf{E}[X] \leq n^{dn}2^{-w/c}$. Therefore, if $w > 4d^2n^3 \log n$, then $\mathbf{E}[X] < 1$. Since X is integral, the latter implies that there exists a universal traversal sequence of length $4d^2n^3 \log n+1$.

Problem 4

This problem deals with investigating the lower and upper bounds on the coupling time of two uncoordinated random walks on undirected, connected and non-bipartite graph G. Where one walker starts from stationary (i.e. uniform on the vertices) and one starts from an arbitrary state.

For an upper-bound on the lower-bound of the coupling time we give as an example complete graph on n vertices with self-loops K_n^*. In this case the coupling time is exactly equal to n, because at each step the two walkers have probability $1/n$ of meeting.

For a (generous) upper-bound, consider a random walk on the product graph $G \times G$, where the walk on the left-side follows the walk on G from stationary, and the walk on the right side follows the walk on G from a fixed arbitrary state. The transition probabilities of this walk on $G \times G$ are exactly uniform across all incident edges for any vertex $v \in V(G \times G)$. Furthermore, the graph is undirected, non-bipartite and d^2-regular. Therefore, tracking the two random walks on G is the same as taking a random walk on $G \times G$.

The two random walks on G meet whenever the walk on $G \times G$ hits a vertex of the form $(v, v) \in G \times G$. The expected time until the latter happens is upper-bounded by the cover time on $G \times G$, which (as shown in class) is no larger than $4|V(G \times G)||E(G \times G)| = 4n^4d^2$.

Problem 5

M is connected because starting from any card ordering $j_1, j_2, \ldots, j_{52}$ we can get to any ordering $i_1, i_2, \ldots, i_{52}$ by applying the Markov transition on the sequence of cards $i_{52}, i_{51}, \ldots, i_1$.

Consider the following coordinator strategy. Pick a random card by its suit and rank (not by its index in the deck). Apply the Markov transition using this card on both decks. Clearly, each deck individually moves along the Markov chain.

After each card has been touched once, the two decks have identical configurations. According to the Coupon collector's problem, this happens on average (and with tight concentration) after $O(n \ln n)$ steps.