LOW-STRETCH SPANNING TREES AND MINIMAX DUALITY

notes by Petar MAYMOUNKOV

1. Notation

We study a connected undirected graph G which is positively edge-weighted by $w(e)$, where a missing edge has $w(e) = \infty$. In way of standard notation, $n = |V(G)|$ and $m = |E(G)|$. We use $\mathbf{1}_Q$ to denote the characteristic vector of the set Q, i.e.

$$\mathbf{1}_Q(i) = \begin{cases} 1, & \text{if } i \in Q \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

and also let $\mathbf{1}$ be the all-ones vector.

2. The graph game

Consider a 2-player 0-sum game with a maximizing *edge player*, denoted x, and a minimizing *tree player*, denoted y. The strategies of the edge player range over the edges $E(G)$, while the strategies of the tree player range over the set $T(G)$ of spanning trees of G. The payoff $a_{(u,v),T}$ at edge $e = (u, v)$ and tree T is defined as

$$a(e, T) := \frac{d_T(u, v)}{w(e)} \tag{2.1}$$

and let $A \in \mathbb{R}^{E(G)\times T(G)}$ be the corresponding payoff matrix. When probabilistic strategies $x \in \mathbb{R}^{E(G)}$ and $y \in \mathbb{R}^{T(G)}$ are fixed, the value of the game is given by

$$x^* A y = \sum_{e,T} x(e) \cdot a(e, T) \cdot y(T) \tag{2.2}$$

In what follows, we are going to use

THEOREM 2.1 (Minimax Principle). —

$$\max_x \min_y x^* A y = \min_y \max_x x^* A y \tag{2.3}$$

Remark 2.2. — Here and throughout we view a vector $z \in \mathbb{R}^Q$ (for some set Q) as a distribution (or, equivalently, a convex combination) over the elements of Q as long as $z \geqslant 0$ and $\|z\|_1 = 1$.

3. Probabilistic stretch implies average stretch

LEMMA 3.1. — *Let $y_\star \in \mathbb{R}^{T(G)}$ be an α-probabilistic approximation of G, then for every challenge distribution $x_{\mathbf{ch}} \in \mathbb{R}^{E(G)}$ there exists a spanning tree $T_\star$ so that*

$$\sum_{e \in E(G)} x_{\mathbf{ch}}(e) \frac{d_{T_\star}(e)}{w(e)} \leqslant \alpha \tag{3.1}$$

Specializing $x_{\mathbf{ch}} = \mathbf{1}_H / |H|$, where $H \subseteq E(G)$, we obtain:

COROLLARY 3.2. — *If G has an α-probabilistic approximation, then for all $H \subseteq E(G)$ there exists a spanning tree $T_\star$ of G with average stretch over the edges in H at most α. Formally,*

$$\frac{1}{|H|} \sum_{e \in H} \frac{d_{T_\star}(e)}{w(e)} \leqslant \alpha \tag{3.2}$$

And when $H = E(G)$ we get the well-known:

COROLLARY 3.3. — *If G has an α-probabilistic approximation, then there exists a spanning tree $T_\star$ of G with average stretch over all edges in G at most α. Formally,*

$$\frac{1}{m} \sum_{e \in E(G)} \frac{d_{T_\star}(e)}{w(e)} \leqslant \alpha \tag{3.3}$$

Proof of Lemma 3.1. — By assumption $y_\star$ is an α-probabilistic approximation of G, i.e. for all $e \in E(G)$ we have

$$\mathbf{E}_{T \sim y_\star} \cdot d_T(e) = \sum_T a(e, T) y_\star(T) \leqslant \alpha, \tag{3.4}$$

equivalently

$$A y_\star \leqslant \alpha \tag{3.5}$$

Thus for any convex combination x

$$x^*(Ay_\star) \leqslant \alpha \tag{3.6}$$

Now, fix a challenge distribution $x_{\mathbf{ch}}$ and observe

$$\begin{aligned}
\min_y x^*_{\mathbf{ch}} Ay &\leqslant \min_y \max_x x^* Ay \qquad (3.7)\\
&= \max_x \min_y x^* Ay \qquad \Big(\text{using Minimax Principle}\Big)\\
&\leqslant \max_x x^* Ay_\star \qquad \Big(\text{since } \min_y x^* Ay \leqslant x^* Ay_\star\Big)\\
&\leqslant \alpha \qquad (3.8) \qquad \Big(\text{using (3.6)}\Big)
\end{aligned}$$

On the other hand, we know that $\min_y x^*_{\mathbf{ch}} Ay$ is achieved at some $y_{\mathbf{ch}} = \mathbf{1}_{T_\star}$ (since y ranges over convex combinations), i.e. (3.8) gives

$$x^*_{\mathbf{ch}} A\mathbf{1}_{T_\star} \leqslant \alpha \tag{3.9}$$

Using the above with (2.2) and (2.1) we obtain

$$\begin{aligned}
x^*_{\mathbf{ch}} A\mathbf{1}_{T_\star} &= \sum_e x_{\mathbf{ch}}(e) a(e, T_\star)\\
&= \sum_e x_{\mathbf{ch}}(e) \frac{d_{T_\star}(e)}{w(e)} \leqslant \alpha.
\end{aligned}$$

∎

4. Lower bound for expanders

THEOREM 4.1. — *All spanning trees of an expander graph family G_n incur average stretch $\Theta(\ln n)$. Thus, expanders have no $o(\ln n)$-probabilistic approximations.*